

Conceptual Aspects of \mathcal{PT} -Symmetry and Pseudo-Hermiticity: A status report

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Abstract. We survey some of the main conceptual developments in the study of \mathcal{PT} -symmetric and pseudo-Hermitian Hamiltonian operators that have taken place during the past ten years or so. We offer a precise mathematical description of a quantum system and its representations that allows us to describe the idea of unitarization of a quantum system by modifying the inner product of the Hilbert space. We discuss the role and importance of the quantum-to-classical correspondence principle that provides the physical interpretation of the observables in quantum mechanics. Finally, we address the problem of constructing an underlying classical Hamiltonian for a unitary quantum system defined by an a priori non-Hermitian Hamiltonian.

1. Introduction

The widespread interest in the study of non-Hermitian but \mathcal{PT} -symmetric Hamiltonian operators such as

$$H = p^2 + i\epsilon x^3, \quad \epsilon \in \mathbb{R}, \quad (1)$$

has its root in the observation that these Hamiltonians can actually possess a real spectrum. This was taken as a sign that perhaps one can relax the usual condition that the Hamiltonian operator (or more generally observables) be Hermitian operators [1]. After all, a non-Hermitian operator such as (1) would define real energy values similarly to a Hermitian operator. This motivated the search for an “extension of quantum mechanics to the complex domain” [2]. There were also claims that this extended quantum theory and its field theoretical generalizations may provide a solution for some of the basic problems of particle physics [3]. Several years of intensive research have however led to a different picture. The purpose of the present article is to give an objective survey of what we have really learned by studying the subject. We will base our treatment on established facts and the basic ideas rather than circumstantial evidence for potential usefulness of the results. This is particularly important for the outsiders who wish to assess the scientific merits of studying the subject and the researchers and students who are undecided whether to join in this effort.

Among the main difficulties one encounters in studying this subject is to make sense of imprecise mathematical statements made in some physics literature and to deal with difficult-to-read mathematical expositions in the relevant mathematics literature. We will try to overcome these difficulties by ignoring the subtle issues that arise whenever the Hilbert space is infinite-dimensional.

Throughout this article we will deal with linear operators mapping between separable complex Hilbert spaces. A separable complex Hilbert space \mathcal{H} is a complex vector space endowed with a positive-definite inner product $\langle \cdot | \cdot \rangle$ such that \mathcal{H} is complete as a metric space and admits a countable basis $\{\xi_n\}$. We denote the dimension of \mathcal{H} by N , and confine our attention to the case that $N < \infty$ unless otherwise is obvious. Our self-imposed restriction to finite-dimensional Hilbert spaces allows us to escape dealing with the technical issues related to the domain of the operators. These issues can be addressed properly using a more careful mathematical analysis that is beyond the scope of the present article. An important observation is that these technical problems and their resolution have no significance as far as the basic conceptual problems of interest are concerned.

We close this section by pointing out that non-Hermitian operators have been the subject of an extensive mathematical research that we have no intention of reviewing in this short report. We refer the interested readers to [4] and references therein. The study of physical applications of non-Hermitian operators has also a long history. The most prominent of these is their crucial role in the description of open quantum systems [5]. There are also numerous applications of these operators in phenomenological/effective descriptions of a variety of physical phenomena. Some of these involve operators with

\mathcal{PT} -symmetry [6]. The present paper does not aim at presenting a general review of non-Hermitian operators and their physical applications. It intends to address very basic and specific questions that arise in trying to employ these operators as Hamiltonians for fundamental, closed, and unitary quantum systems.

2. Hermiticity and self-adjointness

Let \mathcal{V} be an N -dimensional complex vector space with a basis $\{\xi_n\}$, and $H : \mathcal{V} \rightarrow \mathcal{V}$ be a linear operator that is represented by the $N \times N$ matrix \underline{H} in the basis $\{\xi_n\}$. This means that the entries H_{mn} of \underline{H} fulfil

$$H\xi_m = \sum_{n=1}^N H_{nm}\xi_n. \quad (2)$$

\underline{H} is said to be a **Hermitian matrix** if

$$H_{mn}^* = H_{nm}, \quad (3)$$

i.e., $\underline{H}^t = \underline{H}^*$ where superscript t and $*$ stand for transpose and complex-conjugate of \underline{H} , respectively.

Suppose that \underline{H} is a Hermitian matrix. Then it is well-known that it has real eigenvalues and a complete and orthonormal set $\{\vec{e}_n\}$ of eigenvectors.‡ This does not however imply that the operator H possesses the same properties. In fact, we cannot even talk about orthonormality in \mathcal{V} unless we endow it with an inner product.

Now, suppose $\langle \cdot | \cdot \rangle$ is an inner product on \mathcal{V} that makes it into a Hilbert space \mathcal{H} . We can use the basis vectors ξ_n to define a matrix \underline{H}' with the entries $\langle \xi_m | H\xi_n \rangle$. The matrices \underline{H} and \underline{H}' coincide provided that $\{\xi_n\}$ is an orthonormal basis of \mathcal{H} . If this happens to be the case and \underline{H} is a Hermitian matrix, then the operator H does have a real spectrum and a set $\{\psi_n\}$ of eigenvectors ψ_n that forms an orthonormal basis of \mathcal{H} . These are the characteristic properties of self-adjoint operators. By definition $H : \mathcal{H} \rightarrow \mathcal{H}$ is a **self-adjoint operator** if for all $\phi, \psi \in \mathcal{H}$ we have§

$$\langle \phi | H\psi \rangle = \langle H\phi | \psi \rangle. \quad (4)$$

Most textbooks on quantum mechanics follow von Neumann's terminology [7] of using the term “**Hermitian operator**” for self-adjoint operators. Whenever one has a preassigned inner product and uses orthonormal bases to represent linear operators there is no danger of using this terminology, because Hermitian operators are represented by Hermitian matrices. This is the reason why some references identify Hermitian operators with those having Hermitian matrix representations. In the present subject, however, it is absolutely essential not to use a basis-dependent notion such as the Hermiticity

‡ Completeness of $\{\vec{e}_n\}$ means that it is a basis of \mathbb{C}^N . The orthonormality is defined in terms of the Euclidean inner product on \mathbb{C}^N . This is given by $\langle \vec{v} | \vec{w} \rangle_E := \vec{v}^* \cdot \vec{w}$ where the dot stands for the dot product of vectors.

§ Here we ignore the domain issues. For a more precise definition see [8].

of the matrix representation, particularly because the basis one adopts may not be orthonormal with respect to the physically appropriate inner product(s).

Some recent papers use the term “Dirac Hermiticity” of an operator to distinguish the Hermiticity of the matrix representation (in a non-orthonormal basis) and the Hermiticity of the operator [9]. A better solution to this problem is to avoid using matrix representation of operators as much as possible. In particular, we define a Hermitian operator according to (4) rather than (3). The former, basis-independent definition, has the advantage of clarifying the role of the inner product in determining the Hermiticity of a given operator. This is implicit in (3), because this equation implies the Hermiticity of the operator H only if the basis $\{\xi_n\}$ is orthonormal, and this cannot be checked unless one specifies the inner product.||

A similar situation arises in General Relativity where one has the option of using tensor fields, that are covariant quantities, or their component in some coordinate system, that depend on the choice of coordinates. A simple example is the metric tensor \mathbf{g} and its components $g_{\mu\nu}$ in a coordinate system that are linked by $\mathbf{g} = g_{\mu\nu}dx^\mu \otimes dx^\nu$.¶

3. A precise description of quantum systems

Suppose that H is a given linear operator acting in some complex vector space \mathcal{V} and having a real spectrum and a complete set of eigenvectors $\{\psi_n\}$. We wish to address the following central question.

Question 1: Can H serve as the Hamiltonian operator (or more generally an observable) for a quantum system whose state vectors belong to \mathcal{V} ?

Strictly speaking this is an ill-posed question unless we make it clear what we mean by a quantum system. Therefore, we first give a precise definition of a quantum system.

Let \mathcal{H}_1 and $H_1 : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ be a Hilbert space and a linear operator acting in \mathcal{H}_1 that we call a Hamiltonian operator, respectively. Let (\mathcal{H}_2, H_2) be another Hilbert space-Hamiltonian operator pair, and $\langle \cdot | \cdot \rangle_i$ denote the inner product of \mathcal{H}_i for $i = 1, 2$. A linear operator $\mathcal{U} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ with domain \mathcal{H}_1 and range \mathcal{H}_2 is said to be a **unitary operator** if for all $\phi_1, \psi_1 \in \mathcal{H}_1$,

$$\langle \mathcal{U}\phi_1 | \mathcal{U}\psi_1 \rangle_2 = \langle \phi_1 | \psi_1 \rangle_1. \quad (5)$$

We say that (\mathcal{H}_1, H_1) and (\mathcal{H}_2, H_2) are **unitary-equivalent** if there is a unitary operator $\mathcal{U} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ satisfying $\mathcal{U}H_1 = H_2\mathcal{U}$, alternatively $H_2 = \mathcal{U}H_1\mathcal{U}^{-1}$. Unitary-equivalence is an equivalence relation on the set of Hilbert space-Hamiltonian operator pairs. A **quantum system** is an equivalence class⁺ \mathcal{S} of this equivalence relation. Each

|| One may specify the inner product by demanding that $\{\xi_n\}$ be orthonormal. There are certain mathematical subtleties of this procedure when $N = \infty$ [8].

¶ The analogue of this equation that relates a linear operator H to its matrix representation in a basis $\{\xi_n\}$ is $H = \sum_{m,n=1}^N H_{mn}\xi_m \otimes \xi_n^*$, where $\xi_n^* : \mathcal{H} \rightarrow \mathcal{C}$ is the linear map (functional) defined by $\xi_n^*(\sum_{m=1}^N c_m\xi_m) := c_n$, i.e., $\{\xi_n^*\}$ is the dual basis to $\{\xi_n\}$, [10].

⁺ Here an equivalence class is a set of unitary-equivalent Hilbert space-Hamiltonian operator pairs.

Hilbert space-Hamiltonian operator pair belonging to \mathcal{S} is called a representation of \mathcal{S} . A quantum system \mathcal{S} is said to be a **unitary quantum system** if the Hamiltonian operator in all its representations is a Hermitian operator.

What is implicit in the above mathematical description of a quantum system is von-Neumann's axioms of quantum mechanics. According to these axioms, given an arbitrary representation (\mathcal{H}, H) of a quantum system \mathcal{S} , the kinematics and dynamics of a \mathcal{S} are respectively determined by the Hilbert space \mathcal{H} and the Hamiltonian operator H in this representations. In particular, the states and observables of \mathcal{S} in the representation (\mathcal{H}, H) are respectively the rays (one-dimensional subspaces) of \mathcal{H} and certain linear operators $O : \mathcal{H} \rightarrow \mathcal{H}$ acting in \mathcal{H} . The states are uniquely determined by the state vectors $\psi \in \mathcal{H} - \{0\}$ that in general form a dense subset of \mathcal{H} .

The main ingredient of the kinematics of quantum mechanics is von-Neumann's projection (measurement) axiom. Enforcing it puts a strong restriction on the observables. Specifically, it demands that

- (i) the observables O must have a complete set of eigenvectors (for otherwise there may be states of a quantum system that can never be prepared) and
- (ii) for every observable O and state vector ψ , the expectation value, $\langle \psi | O \psi \rangle / \langle \psi | \psi \rangle$ is a real number. This is because the results of measurements and their expected values are real numbers.

It is this requirement of the reality of expectation values that forces observables to be Hermitian (self-adjoint) operators acting in \mathcal{H} . This is a direct consequences of a well-known mathematical theorem that is unfortunately not discussed in standard textbooks on quantum mechanics.* In view of this theorem the reality of the spectrum of an operator is only a necessary condition for a consistent implementation of the projection axiom. The Hermiticity of the observables, however, is both necessary and sufficient. This shows that unless one wishes to modify the projection axiom one cannot escape the condition of the Hermiticity of observables. This also applies to the Hamiltonian operator even at the kinematic level, if one demands that it is also an observable of the quantum system.

The dynamics of the quantum system \mathcal{S} in a representation (\mathcal{H}, H) is described by the Hamiltonian operator H through the Schrödinger (or Heisenberg) equation. Again consistency of dynamics with projection axiom demands that the evolution of the state vectors $\psi(t_0) \rightarrow \psi(t) = U(t, t_0)\psi(t_0)$ is affected by a unitary operator $U(t, t_0)$. In view of the Schrödinger equation: $i\hbar \frac{d}{dt}U(t, t_0) = HU(t, t_0)$, this also implies that H is a Hermitian operator.† Therefore, a consistent application of the projection axiom alone demands that the quantum system must be unitary.

The von-Neumann axioms of quantum mechanics are valid in all of the representations of a unitary quantum system. The choice of the representation, which

* See for example the appendix of [8].

† This is known as the Stone's theorem [11].

is clearly not unique, depends on the observer. The freedom of making this choice is similar to an observer's freedom to choose a particular unit system. Clearly, the physical quantities associated with the quantum system \mathcal{S} are independent of the choice of a representation. This can be shown to be a consequence of the unitary-equivalence of the representations. For example let \mathcal{S} be in a state described by $\psi_1 \in \mathcal{H}_1 - \{0\}$ in a representation (\mathcal{H}_1, H_1) and O_1 be an observable in this representation. The expectation value of O_1 in this state is given by $\langle \psi_1 | O_1 \psi_1 \rangle_1 / \langle \psi_1 | \psi_1 \rangle_1$. Now, let (\mathcal{H}_2, H_2) be another representation of \mathcal{S} . Then there is a unitary operator $\mathcal{U} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ that maps ψ_1 and O_1 to $\psi_2 := \mathcal{U}\psi_1$ and $O_2 := \mathcal{U}O_1\mathcal{U}^{-1}$, respectively. In view of (5), $\langle \psi_2 | O_2 \psi_2 \rangle_2 / \langle \psi_2 | \psi_2 \rangle_2 = \langle \psi_1 | O_1 \psi_1 \rangle_1 / \langle \psi_1 | \psi_1 \rangle_1$. This shows that the expectation values are representation-independent.

The above discussion of quantum systems and their representations provides a complete answer for Question 1, namely H can serve as the Hamiltonian operator for a quantum system \mathcal{S} represented by (\mathcal{H}, H) provided that it is a Hermitian operator acting in \mathcal{H} . In particular completeness of the eigenvectors and reality of the spectrum of H are necessary but not sufficient. However, it turns out that if H is not Hermitian but possesses these two properties, then it can serve as the Hamiltonian operator for another quantum system. As a result of a theorem established in [12], *if H has a real spectrum and a complete set of eigenvectors, one can modify the inner product of \mathcal{H} to define a new Hilbert space \mathcal{H}' in such a way that as a linear operator acting in \mathcal{H}' , H is a Hermitian operator.* In this way (\mathcal{H}', H) represents a unitary quantum system that we denote by \mathcal{S}' . Refs. [12, 13, 14] give a construction of the modified inner product that defines \mathcal{H}' and consequently \mathcal{S}' . For a comprehensive review of this construction and related developments, see [8].

4. Pseudo-Hermiticity and Antilinear Symmetries

In the preceding section we gave a complete answer to Question 1. But this answer did not involve a discussion of \mathcal{PT} -symmetry that has in a sense become a landmark of the subject. This motivates the following question.

Question 2: How essential is \mathcal{PT} -symmetry?

To understand the relation between \mathcal{PT} -symmetry and the idea of modifying the inner product of the Hilbert space we need to recall some basic mathematical notions.

A linear operator $\eta : \mathcal{H} \rightarrow \mathcal{H}$ is called a **pseudo-metric operator** if it is a Hermitian automorphism, i.e., it is a Hermitian one-to-one linear operator having \mathcal{H} both as its domain and range. A **metric operator** is a positive-definite pseudo-metric operator.

We say that a linear operator $H : \mathcal{H} \rightarrow \mathcal{H}$ is **pseudo-Hermitian** if there is a pseudo-metric operator $\eta : \mathcal{H} \rightarrow \mathcal{H}$ satisfying^{††}

$$H^\dagger = \eta H \eta^{-1}. \quad (6)$$

Suppose we are given a particular pseudo-metric operator $\eta : \mathcal{H} \rightarrow \mathcal{H}$. Then a linear operator H satisfying (6) is called **η -pseudo-Hermitian**.

Given a pseudo-Hermitian operator H , one can choose one of the pseudo-metric operators η satisfying (6) to construct a pseudo-inner product according to

$$\langle \phi | \psi \rangle_\eta := \langle \phi | \eta \psi \rangle, \quad (7)$$

where $\langle \cdot | \cdot \rangle$ is the inner product of \mathcal{H} . If η happens to be a positive-definite operator, then $\langle \cdot | \cdot \rangle_\eta$ is a genuine positive-definite inner product and we can use it to define a Hilbert space \mathcal{H}' in which H acts as a Hermitian operator. If a positive-definite η fulfilling (6) exists, H is called **quasi-Hermitian** [15].

It turns out that a necessary and sufficient condition for pseudo-Hermiticity of a linear operator with a complete set of eigenvectors is that it commutes with an invertible antilinear operator [13, 16]. In particular, H is quasi-Hermitian if and only if it has a complete set of common eigenvectors with an invertible antilinear operator. This is the link to \mathcal{PT} -symmetry. Because \mathcal{PT} is just a particular example of an invertible antilinear operator, \mathcal{PT} -symmetric quasi-Hermitian Hamiltonians operators constitute a special class of quasi-Hermitian operators. This shows that indeed \mathcal{PT} -symmetry is not an essential ingredient of the subject. One can easily construct quasi-Hermitian Hamiltonian operators that possess other types of invertible antilinear symmetries [17]. One can apply the procedure of defining a unitary quantum system by modifying the inner product of the Hilbert space using these Hamiltonian operators. Typical examples are the complex point interactions [18, 19].

5. Correspondence principle and classical limit

In section 3, we outlined a mathematical description of a quantum system \mathcal{S} in terms of unitary-equivalence classes of Hilbert space-Hamiltonian operator pairs (\mathcal{H}, H) . An important aspect of this formulation is the procedure according to which we assign a physical meaning to observables. This is essentially based on the quantum-to-classical correspondence principle.

Consider a typical quantum system represented by (\mathcal{H}_0, H) where \mathcal{H}_0 is the space of square-integrable functions, $L^2(\mathbb{R}) := \{\psi : \mathbb{R} \rightarrow \mathbb{C} \mid \int_{\mathbb{R}} |\psi(x)|^2 dx < \infty\}$, endowed with the standard L^2 -inner product, $\langle \phi | \psi \rangle := \int_{\mathbb{R}} \phi(x)^* \psi(x) dx$. It is customary to take the operators $\hat{x}, \hat{p} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ defined by

$$(\hat{x}\psi)(x) = x\psi(x), \quad (\hat{p}\psi)(x) = -i\hbar \frac{d}{dx} \psi(x), \quad (8)$$

^{††}Here H^\dagger denotes the adjoint of H that is defined by the condition: $\langle \phi | H \psi \rangle = \langle H^\dagger \phi | \psi \rangle$ for all $\phi, \psi \in \mathcal{H}$. For a more general and precise definition of the adjoint operator, see [8].

as the position and momentum observables in this representation. The assignment of the physical meaning of “position” and “momentum” to purely mathematical entities such as \hat{x} and \hat{p} is a manifestation of the quantum-to-classical correspondence principle/or the canonical quantization scheme. For the case we consider, this principle assigns the operators \hat{x} and \hat{p} to the classical observables of position and momentum of a particle moving on the real line. It is absolutely essential that this assignment is consistent with the correspondence of the Poisson brackets with commutators: $\{\cdot, \cdot\} \leftrightarrow (i\hbar)^{-1}[\cdot, \cdot]$.

Now, consider modifying the L^2 -inner product as follows. Let $\eta := e^{-\kappa\mathcal{P}}$ where $\kappa \in \mathbb{R}$ and \mathcal{P} is the parity operator: $(\mathcal{P}\psi)(x) := \psi(-x)$. η is a metric operator and $\langle \cdot | \cdot \rangle_\eta$ is a positive-definite inner product on $L^2(\mathbb{R})$. It is easy to check that the free particle Hamiltonian $H_0 := \hat{p}^2/(2m)$ is η -pseudo-Hermitian. Therefore, if we define \mathcal{H}' by endowing $L^2(\mathbb{R})$ with the inner product $\langle \cdot | \cdot \rangle_\eta$, we find a quantum system represented by (\mathcal{H}', H_0) . As operators acting in \mathcal{H}' , \hat{x} and \hat{p} are not Hermitian. In particular, there is no justification for calling them position and momentum of a particle. What plays the role of \hat{x} and \hat{p} in \mathcal{H}' are the operators: $\hat{x}' := e^{\kappa\mathcal{P}}\hat{x}$ and $\hat{p}' := e^{\kappa\mathcal{P}}\hat{p}$, [20]. In other words the quantum-to-classical correspondence principle takes the form:

$$x \leftrightarrow \hat{x}', \quad x \leftrightarrow \hat{x}', \quad \{\cdot, \cdot\} \leftrightarrow (i\hbar)^{-1}[\cdot, \cdot]. \quad (9)$$

It is not difficult to show that in fact (\mathcal{H}_0, H_0) and (\mathcal{H}', H_0) are unitary-equivalent. Therefore (\mathcal{H}', H_0) is just another equally admissible representation of the quantum system consisting of a free particle moving on \mathbb{R} .

Now, consider a more general case where (\mathcal{H}_0, H) does not represent a unitary quantum system, but modifying the inner product of $L^2(\mathbb{R})$ we obtain a Hilbert space \mathcal{H}' such that (\mathcal{H}', H) represents a unitary quantum system \mathcal{S} . A typical example is the \mathcal{PT} -symmetric Hamiltonian (1). In this case we cannot assign any physical meaning to Hermitian operators acting in \mathcal{H}_0 . Rather, we need to construct Hermitian operators acting in \mathcal{H}' and set up a correspondence between these and the classical observables. For the explicit form of the position and momentum operators associated with the unitary quantum systems that are determined by the Hamiltonian (1), see [20].

Once the operators \hat{x}' and \hat{p}' associated with position and momentum observables in the representation (\mathcal{H}', H) are determined, we can express the Hamiltonian operator H in terms of \hat{x}' and \hat{p}' and take the classical limit:

$$\hat{x}' \rightarrow x, \quad \hat{p}' \rightarrow p, \quad \hbar \rightarrow 0. \quad (10)$$

This yields the underlying classical Hamiltonian for the unitary quantum system \mathcal{S} . It is via this procedure that we can give a physical meaning to H and consequently \mathcal{S} .

A more convenient method of determining the underlying classical Hamiltonian is by constructing a Hermitian Hamiltonian operator h acting in \mathcal{H}_0 such that (\mathcal{H}_0, h) is unitary-equivalent to (\mathcal{H}', H) . We can obtain h using the following formula provided that we are given a metric operator η such that H is η -pseudo-Hermitian.

$$h := \eta^{1/2} H \eta^{-1/2}. \quad (11)$$

h is a Hermitian operator acting in \mathcal{H}_0 , because $\eta^{1/2} : \mathcal{H}' \rightarrow \mathcal{H}_0$ is a unitary operator, [21]. Having obtained the representation (\mathcal{H}_0, h) we have the standard choice for position and momentum operators and can identify the underlying classical Hamiltonian by expression h in terms of \hat{x} and \hat{p} and taking the usual classical limit:

$$\hat{x} \rightarrow x, \quad \hat{p} \rightarrow p, \quad \hbar \rightarrow 0. \quad (12)$$

The existence of the representation (\mathcal{H}_0, h) seems to indicate that we can completely avoid the use of nonstandard inner products and apply the standard methods of quantum mechanics to describe the quantum system \mathcal{S} . This is true in principle, but extremely difficult to implement in practice. The reason is that unlike H , the equivalent Hermitian Hamiltonian h is, in general, a highly nonlocal (integral) operator.

In Ref. [22], the authors make another proposal for assigning an underlying classical system for the unitary quantum systems defined by the quasi-Hermitian Hamiltonians such as (1). This involves implementing the usual classical limit (12) in the expression for H directly and imposing the Hamilton's classical equations of motion that correspond to the resulting classical Hamiltonian \mathfrak{H} . The main difficulty with this approach is that \mathfrak{H} is a complex-valued function of x and p . As a results, the Hamilton's equations define a classical dynamical system in the complex phase space \mathbb{C}^2 rather than the real phase space \mathbb{R}^2 (that have (x, p) as its coordinates.)

A careful study of the structure of this complex dynamical system reveals that its dynamics is not consistent with the usual Poisson bracket (symplectic structure) on the phase space $\mathbb{C}^2 = \mathbb{R}^4$. To assure the dynamical-kinematical consistency of the description of this system, one is forced to endow the phase space $\mathbb{C}^2 = \mathbb{R}^4$ with a modified Poisson bracket [23, 24]. It turns that these complex classical systems also admit a real description and using this real description one discovers that they are completely integrables systems possessing a specific gauge symmetry [24, 25].

A major problem with identifying these complex classical dynamical systems with the classical counterparts of the original unitary quantum systems is the lack of an explicit quantum-to-classical correspondence (such as (9)). In fact, such a correspondence cannot be implemented directly, because the quantum system has a two- (real) dimensional phase space whereas the complex classical system has a four- (real) dimensional phase space. One may attempt to reduce the phase space to a two-dimensional subspace by fixing a gauge. This has so far not led to a desired correspondence between quantum and classical observables. More problematic is that even for non-unitary quantum systems one can apply the same method and obtain a complex classical system.

Recently Bender et al [26] have tried to restrict this complex classical dynamics to certain contours in the complex x -plane and introduce a real, positive, and integrable function on these contours that they propose to identify with a probability density. In the opinion of the present author, one cannot begin to speak about a probability density before making it clear which observable one is measuring and what kind of a measurement axiom one adopts. All this cannot be establish before one devises a

correspondence rule between classical and quantum observables.

6. Concluding Remarks

The advent of non-Hermitian \mathcal{PT} -symmetric Hamiltonians with a real spectrum led to the expectations that one can indeed extend quantum mechanics to a more general physical theory in which such Hamiltonian operators can also be used to model fundamental unitary quantum systems. Our current understanding is that we achieve the latter goal not by modifying or extending quantum mechanics as a physical theory but by using alternative representations of quantum systems where the Hilbert space is defined by a nonstandard inner product. This is realized within the confines of the standard quantum mechanics provided that we give a precise and sufficiently general definition of a quantum system.

It turns out that actually \mathcal{PT} -symmetry does not play an essential role in implementing this idea. It serves as a particular manifestation of the mathematical fact that every linear operator that is capable of serving as the Hamiltonian operator H for a unitary quantum system commutes with an invertible antilinear operator \mathfrak{S} . In fact, H and \mathfrak{S} share a common complete set of eigenvectors.

The unitary quantum systems defined through the modification of the inner product of the Hilbert space cannot be given a physical interpretation unless one specifies their underlying classical Hamiltonian. We outlined the existing methods of achieving this and commented on the necessity of modifying the symplectic structure on the phase space of the complex dynamical systems obtained by taking the standard classical limit of the non-Hermitian Hamiltonians.

We conclude by emphasizing that the main problems related with the conceptual and structural aspects of the subject have more or less been resolved. What remains to be investigated are the concrete physical applications of the results. Among interesting developments in this direction are the applications in relativistic quantum mechanics [27, 28], quantum cosmology [29], quantum field theory [30], bound-state scattering [31], and electromagnetic wave propagation [32].

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